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Periodic solutions of a nonautonomous predator–prey system with stage structure and time delays

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Abstract

A nonautonomous Lotka–Volterra type predator–prey model with stage structure and time delays is investigated. It is assumed in the model that the individuals in each species may belong to one of two classes: the immatures and the matures, the age to maturity is presented by a time delay, and that the immature predators do not feed on prey and do not have the ability to reproduce. By some comparison arguments we first discuss the permanence of the model. By using the continuation theorem of coincidence degree theory, sufficient conditions are derived for the existence of positive periodic solutions to the model. By means of a suitable Lyapunov functional, sufficient conditions are obtained for the uniqueness and global stability of the positive periodic solutions to the model.

MSC: Primary 34K13; 34K60; 92D25

Keywords: Stage structure; Time delay; Permanence; Periodic solution; Global stability

1. Introduction

The aim of this paper is to investigate the effects of introducing *stage structure* to the dynamics of a Lotka–Volterra type predator–prey model. Stage structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. As is common, the dynamics—eating habits, susceptibility to predators, etc.—are often quite different in these two sub-populations. Hence, it is of ecological importance to investigate the effects of such a subdivision on the interaction of species.

Population models with stage structure are of current research interest in mathematical biology. They can exhibit phenomena similar to those of partial differential equations and many important physiological parameters can be incorporated [3]. Moreover, they are often much simpler than the corresponding models governed by partial differential equations. There has been much work on modelling stage-structured population models (see, for example, [1–3,5,8–21]). In [1], the work of Aiello and Freedman on a single species growth model with stage structure, represents a mathematically careful and biologically meaningful approach. In [1], a model of single species population growth incorporating

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stage structure as a reasonable generalization of the classical logistic model was derived and investigated. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. The model takes the form

$$\begin{aligned}&\dot{x}_i(t)=\alpha x_m(t)-\gamma x_i(t)-\alpha \mathrm{e}^{-\gamma\tau}x_m(t-\tau),\\&\dot{x}_m(t)=\alpha \mathrm{e}^{-\gamma\tau}x_m(t-\tau)-\beta x_m^2(t),\quad t>\tau,\end{aligned}\tag{1.1}$$

where $x_i(t)$ denotes the immature population density, $x_m(t)$ represents the mature population density, $\alpha>0$ represents the birth rate, $\gamma>0$ is the immature death rate, $\beta>0$ is the mature death and overcrowding rate, τ is the time to maturity. The term $\alpha \mathrm{e}^{-\gamma\tau}x_m(t-\tau)$ represents the immatures who were born at time $t-\tau$ and survive at time t (with the immature death rate γ), and therefore represents the transformation of immatures to matures.

We note that any biological or environmental parameters are naturally subject to fluctuation in time. As Cushing [4] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment.

Motivated by the recent work of Aiello and Freedman [1], in this paper we study the effects of stage structure for both prey and predator populations and the periodicity of ecological and environmental parameters on the global dynamics of a Lotka–Volterra type predator–prey system. To do so, we study the following delayed differential system

$$\begin{cases}\dot{x}_1(t)=\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t\gamma_1(s)\,\mathrm{d}s}x_1(t-\tau_1)-a_{11}(t)x_1^2(t)-a_{12}(t)x_1(t)x_2(t),\\ \dot{y}_1(t)=\alpha_1(t)x_1(t)-\gamma_1(t)y_1(t)-\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t\gamma_1(s)\,\mathrm{d}s}x_1(t-\tau_1),\\ \dot{x}_2(t)=\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t\gamma_2(s)\,\mathrm{d}s}x_1(t-\tau_2)x_2(t-\tau_2)-r(t)x_2(t)-a_{22}(t)x_2^2(t),\\ \dot{y}_2(t)=\alpha_2(t)x_1(t)x_2(t)-\gamma_2(t)y_2(t)-\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t\gamma_2(s)\,\mathrm{d}s}x_1(t-\tau_2)x_2(t-\tau_2),\end{cases}\tag{1.2}$$

where $x_1(t)$ and $y_1(t)$ denote the densities of the mature and immature prey populations at time t, respectively, $x_2(t)$ and $y_2(t)$ represent the densities of the mature and immature predator populations at time t, respectively. $a_{11}(t), a_{12}(t), a_{22}(t), r(t), \alpha_1(t), \alpha_2(t), \gamma_1(t), \gamma_2(t)$ are continuously positive ω-periodic functions, τ_1 and τ_2 are positive constants. The model is derived under the following assumptions.

(A1) *The prey population*: The birth rate is proportional to the existing mature population with a proportionality $\alpha_1(t)>0$; the death rate of the immature population is proportional to the existing immature population with a proportionality $\gamma_1(t)>0$; $a_{11}(t)$ is the death and intra-specific competition rate of the mature population. The term $\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t\gamma_1(s)\,\mathrm{d}s}x_1(t-\tau_1)$ represents the immature prey individuals who were born at time $t-\tau_1$ and survive at time t, and therefore represents the transformation of immature prey population to mature prey population. We refer to the recent article of Liu et al. [11].

(A2) *The predator population*: $a_{12}(t)$ is the capturing rate of the mature predators, $\alpha_2(t)/a_{12}(t)$ is the conversion rate of nutrients into the reproduction of the mature predators, $r(t)$ is the death rate of the mature predators, $a_{22}(t)$ is the intra-specific competition rate of the mature predators, the death rate of the immature population is proportional to the existing immature population with a proportionality $\gamma_2(t)>0$. The term $\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t\gamma_2(s)\,\mathrm{d}s}x_1(t-\tau_2)x_2(t-\tau_2)$ represents the number of immature predators that were born at time $t-\tau_2$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. It is assumed in (1.2) that immature individual predators do not feed on prey and do not have the ability to reproduce.

The initial conditions for system (1.2) take the form

$$\begin{aligned}&x_i(\theta)=\phi_i(\theta),\quad y_i(\theta)=\psi_i(\theta),\\&\phi_i(0)>0,\quad \psi_i(0)>0,\quad i=1,2,\end{aligned}\tag{1.3}$$

where $(\phi_1(\theta),\psi_1(\theta),\phi_2(\theta),\psi_2(\theta))\in C([-\tau,0],R_{+0}^4)$, the Banach space of continuous functions mapping the interval $[-\tau,0]$ into R_{+0}^4, where $\tau=\max\{\tau_1,\tau_2\}$, $R_{+0}^4=\{(x_1,x_2,x_3,x_4)|x_i\geqslant 0, i=1,2,3,4\}$.

For continuity of the initial conditions, we further require

$$y_1(0) = \int_{-\tau_1}^{0} \alpha_1(s) e^{-\int_s^0 \gamma_1(v)\,dv} \phi_1(s)\,ds,$$
$$y_2(0) = \int_{-\tau_2}^{0} \alpha_2(s) e^{-\int_s^0 \gamma_2(v)\,dv} \phi_1(s)\phi_2(s)\,ds. \tag{1.4}$$

We adopt the following notations throughout this paper:

$$\bar{f} = \frac{1}{\omega}\int_0^{\omega} f(t)\,dt, \quad f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{[0,\omega]} f(t),$$

where f is a continuous ω-periodic function.

The paper is organized as follows. In the next section, we discuss the positivity of solutions and the permanence of system (1.2). In Section 3, sufficient conditions are derived for the existence of positive periodic solutions of system (1.2) by using Gaines and Mawhin's continuation theorem of coincidence degree theory. In Section 4, by means of a suitable Lyapunov functional, we are concerned with the uniqueness and global stability of the positive periodic solutions to system (1.2). A brief discussion is presented in Section 5 to conclude this work.

2. Permanence

In this section, we are concerned with the permanence of system (1.2). To do so, we first need to show the positivity of solutions of system (1.2) with initial conditions (1.3)–(1.4).

Lemma 2.1. *Solutions of system* (1.2) *with initial conditions* (1.3) *and* (1.4) *are positive for all* $t \geqslant 0$.

Proof. Let $(x_1(t), y_1(t), x_2(t), y_2(t))$ be a solution of system (1.2) with initial conditions (1.3) and (1.4). Let us first consider $x_2(t)$ for $t \in [0, \tau^*]$, where $\tau^* = \min\{\tau_1, \tau_2\}$. We derive from the third equation of system (1.2) that

$$\begin{aligned}\dot{x}_2(t) &= \alpha_2(t-\tau_2) e^{-\int_{t-\tau_2}^{t} \gamma_2(s)\,ds} \phi_1(t-\tau_2)\phi_2(t-\tau_2) - r(t)x_2(t) - a_{22}(t)x_2^2(t)\\ &\geqslant x_2(t)[-r(t) - a_{22}(t)x_2(t)]\end{aligned}$$

since $\phi_1(\theta)\geqslant 0,\ \ \phi_2(\theta)\geqslant 0$ for $\theta \in [-\tau, 0]$. By comparison, it follows that for $t \in [0, \tau^*]$,

$$x_2(t) \geqslant \frac{x_2(0)\ \exp[-\int_0^t r(s)\,ds]}{1 + x_2(0)\int_0^t a_{22}(s)\ \exp[-\int_0^s r(u)\,du]\,ds} > 0.$$

Noting that $\phi_1(\theta)\geqslant 0$, $\theta \in [-\tau, 0]$, we derive from the first equation of system (1.2) that for $t \in [0, \tau^*]$,

$$\begin{aligned}\dot{x}_1(t) &= \alpha_1(t-\tau_1) e^{-\int_{t-\tau_1}^{t} \gamma_1(s)\,ds} \phi_1(t-\tau_1) - a_{11}(t)x_1^2(t) - a_{12}(t)x_1(t)x_2(t)\\ &\geqslant x_1(t)[-a_{12}(t)x_2(t) - a_{11}(t)x_1(t)].\end{aligned}$$

By comparison, it follows that for $t \in [0, \tau^*]$,

$$x_1(t) \geqslant \frac{x_1(0)\ \exp[-\int_0^t (a_{12}(s)x_2(s))\,ds]}{1 + x_1(0)\int_0^t a_{11}(s)\ \exp[-\int_0^s a_{12}(u)x_2(u)\,du]\,ds} > 0.$$

In a similar way, we treat the intervals $[\tau^*, 2\tau^*], \ldots, [n\tau^*, (n+1)\tau^*]$, $n \in N$. Thus, $x_1(t) > 0$ and $x_2(t) > 0$ for all $t \geqslant 0$.

We derive from (1.2) and (1.4) that

$$y_1(t) = \int_{t-\tau_1}^{t} \alpha_1(s) \mathrm{e}^{-\int_s^t \gamma_1(v)\,\mathrm{d}v} x_1(s)\,\mathrm{d}s,$$
$$y_2(t) = \int_{t-\tau_2}^{t} \alpha_2(s) \mathrm{e}^{-\int_s^t \gamma_2(v)\,\mathrm{d}v} x_1(s) x_2(s)\,\mathrm{d}s. \tag{2.1}$$

Therefore, the positivity of $y_1(t)$ and $y_2(t)$ follows. The proof is complete. □

In order to discuss the permanence of system (1.2), we need the following result from [16].

Lemma 2.2. *Consider the following equation*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t),$$

where a, b, c *and* τ *are positive constants*, $x(t) > 0$ *for* $t \in [-\tau, 0]$. *We have*

(i) *If* $a > b$, *then* $\lim_{t\to+\infty} x(t) = (a-b)/c$;
(ii) *If* $a < b$, *then* $\lim_{t\to+\infty} x(t) = 0$.

We are now able to state and prove our result on the permanence of system (1.2) with initial conditions (1.3) and (1.4).

Theorem 2.1. *System* (1.2) *with initial conditions* (1.3) *and* (1.4) *is permanent provided that*

(H1) $a_{11}^L a_{22}^L (\alpha_1^L \alpha_2^L \mathrm{e}^{-\gamma_1^M \tau_1 - \gamma_2^M \tau_2} - r^M a_{11}^M) > a_{12}^M \alpha_2^L \mathrm{e}^{-\gamma_2^M \tau_2} (\alpha_1^M \alpha_2^M \mathrm{e}^{-\gamma_1^L \tau_1 - \gamma_2^L \tau_2} - r^L a_{11}^L) > 0.$

Proof. Let $(x_1(t), y_1(t), x_2(t), y_2(t))$ be a positive solution of system (1.2) with initial conditions (1.3) and (1.4).

We first give the upper bounds of the positive solution of system (1.2). It follows from the first equation of system (1.2) that

$$\dot{x}_1(t) \leqslant \alpha_1^M \mathrm{e}^{-\gamma_1^L \tau_1} x_1(t-\tau_1) - a_{11}^L x_1^2(t).$$

Consider the following auxiliary equation

$$\dot{u}(t) = \alpha_1^M \mathrm{e}^{-\gamma_1^L \tau_1} u(t-\tau_1) - a_{11}^L u^2(t).$$

By Lemma 2.2 we obtain that

$$\lim_{t\to+\infty} u(t) = \frac{\alpha_1^M \mathrm{e}^{-\gamma_1^L \tau_1}}{a_{11}^L}.$$

By comparison it follows that

$$\limsup_{t\to+\infty} x_1(t) \leqslant \frac{\alpha_1^M \mathrm{e}^{-\gamma_1^L \tau_1}}{a_{11}^L}.$$

Therefore, for $\varepsilon > 0$ sufficiently small there is a $T_{11} > 0$ such that if $t > T_{11}$,

$$x_1(t) \leqslant \frac{\alpha_1^M \mathrm{e}^{-\gamma_1^L \tau_1}}{a_{11}^L} + \varepsilon := M_1. \tag{2.2}$$

We derive from the third equation of system (1.2) for $t > T_{11} + \tau$ that

$$\dot{x}_2(t) \leqslant \alpha_2^M \mathrm{e}^{-\gamma_2^L \tau_2} \left(\frac{\alpha_1^M \mathrm{e}^{-\gamma_1^L \tau_1}}{a_{11}^L} + \varepsilon \right) x_2(t-\tau_2) - r^L x_2(t) - a_{22}^L x_2^2(t).$$

By comparison it follows that

$$\limsup_{t\to+\infty} x_2(t) \leqslant \frac{\alpha_2^M e^{-\gamma_2^L\tau_2}((\alpha_1^M e^{-\gamma_1^L\tau_1}/a_{11}^L)+\varepsilon)-r^L}{a_{22}^L}.$$

Since $\varepsilon>0$ is arbitrary and sufficiently small, we can conclude that

$$\limsup_{t\to+\infty} x_2(t) \leqslant \frac{\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^La_{11}^L}{a_{11}^La_{22}^L}.$$

Hence, for $\varepsilon>0$ sufficiently small there exists a $T_{12}>T_{11}+\tau$ such that if $t>T_{12}$,

$$x_2(t)\leqslant \frac{\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^La_{11}^L}{a_{11}^La_{22}^L}+\varepsilon := M_2. \tag{2.3}$$

Setting $T_1=T_{12}+\tau$, it follows from (2.1) that for $t>T_1$,

$$y_1(t)\leqslant \frac{\alpha_1^M M_1}{\gamma_1^L}(1-e^{-\gamma_1^L\tau_1}),\qquad y_2(t)\leqslant\frac{\alpha_2^M M_1M_2}{\gamma_2^L}(1-e^{-\gamma_2^L\tau_2}).$$

We now show the lower bounds of positive solutions to system (1.2). It follows from the first equation of system (1.2) and (2.3) that for $t>T_1$,

$$\dot{x}_1(t)\geqslant \alpha_1^L e^{-\gamma_1^M\tau_1}x_1(t-\tau_1)-a_{11}^Mx_1^2(t)-a_{12}^MM_2x_1(t).$$

Consider the following auxiliary equation

$$\dot{u}(t)=\alpha_1^L e^{-\gamma_1^M\tau_1}u(t-\tau_1)-a_{12}^MM_2u(t)-a_{11}^Mu^2(t).$$

By Lemma 2.2 we obtain that

$$\lim_{t\to+\infty} u(t)=\frac{\alpha_1^Le^{-\gamma_1^M\tau_1}-a_{12}^MM_2}{a_{11}^M}.$$

A standard comparison argument shows that

$$\liminf_{t\to+\infty} x_1(t)\geqslant\frac{\alpha_1^Le^{-\gamma_1^M\tau_1}-a_{12}^M((\alpha_1^M\alpha_2^Me^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^La_{11}^L/a_{11}^La_{22}^L)+\varepsilon)}{a_{11}^M}.$$

Since $\varepsilon>0$ is arbitrary and sufficiently small, we conclude that

$$\liminf_{t\to+\infty} x_1(t)\geqslant\frac{\alpha_1^Le^{-\gamma_1^M\tau_1}-(a_{12}^M/a_{11}^La_{22}^L)(\alpha_1^M\alpha_2^Me^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^La_{11}^L)}{a_{11}^M}.$$

Therefore, for $\varepsilon>0$ sufficiently small there is a $T_2>T_1$ such that if $t>T_2$,

$$x_1(t)>\frac{\alpha_1^Le^{-\gamma_1^M\tau_1}-(a_{12}^M/a_{11}^La_{22}^L)(\alpha_1^M\alpha_2^Me^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^La_{11}^L)}{a_{11}^M}-\varepsilon := m_1. \tag{2.4}$$

It follows from the third equation of system (1.2) and (2.4) that for $t>T_2+\tau$,

$$\dot{x}_2(t)\geqslant\alpha_2^Le^{-\gamma_2^M\tau_2}m_1x_2(t-\tau_2)-r^Mx_2(t)-a_{22}^Mx_2^2(t).$$

Consider the following auxiliary equation

$$\dot{u}(t)=\alpha_2^Le^{-\gamma_2^M\tau_2}m_1u(t-\tau_2)-r^Mu(t)-a_{22}^Mu^2(t).$$

By Lemma 2.2 we derive that

$$\begin{aligned}
&\lim_{t\to+\infty} u(t)\\
&\quad=\frac{1}{a_{22}^M}\left\{\alpha_2^L \mathrm{e}^{-\gamma_2^M\tau_2}\left(\frac{\alpha_1^L \mathrm{e}^{-\gamma_1^M\tau_1}-(a_{12}^M/a_{11}^L a_{22}^L)(\alpha_1^M\alpha_2^M \mathrm{e}^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^L a_{11}^L)}{a_{11}^M}-\varepsilon\right)-r^M\right\}.
\end{aligned}$$

By comparison it follows that

$$\begin{aligned}
&\liminf_{t\to+\infty} x_2(t)\\
&\quad\geqslant\frac{1}{a_{22}^M}\left\{\alpha_2^L \mathrm{e}^{-\gamma_2^M\tau_2}\left(\frac{\alpha_1^L \mathrm{e}^{-\gamma_1^M\tau_1}-(a_{12}^M/a_{11}^L a_{22}^L)(\alpha_1^M\alpha_2^M \mathrm{e}^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^L a_{11}^L)}{a_{11}^M}-\varepsilon\right)-r^M\right\}.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary and sufficiently small, we can conclude that

$$\begin{aligned}
\liminf_{t\to+\infty} x_2(t)\geqslant\frac{1}{a_{11}^M a_{22}^M}&\left\{\alpha_1^L\alpha_2^L \mathrm{e}^{-\gamma_1^M\tau_1-\gamma_2^M\tau_2}-r^M a_{11}^M\right.\\
&\left.-\frac{a_{12}^M\alpha_2^L \mathrm{e}^{-\gamma_2^M\tau_2}}{a_{11}^L a_{22}^L}(\alpha_1^M\alpha_2^M \mathrm{e}^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^L a_{11}^L)\right\}.
\end{aligned}$$

Hence, for $\varepsilon>0$ sufficiently small there is a $T_3>T_2+\tau$ such that if $t>T_3$,

$$\begin{aligned}
x_2(t)>\frac{1}{a_{11}^M a_{22}^M}&\left\{\alpha_1^L\alpha_2^L \mathrm{e}^{-\gamma_1^M\tau_1-\gamma_2^M\tau_2}-r^M a_{11}^M\right.\\
&\left.-\frac{a_{12}^M\alpha_2^L \mathrm{e}^{-\gamma_2^M\tau_2}}{a_{11}^L a_{22}^L}(\alpha_1^M\alpha_2^M \mathrm{e}^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}-r^L a_{11}^L)\right\}-\varepsilon:=m_2.
\end{aligned}\tag{2.5}$$

We note that if (H1) holds and $\varepsilon>0$ is sufficiently small, $m_i>0$ $(i=1,2)$.

We derive from (2.1), (2.4) and (2.5) that there is a $T>T_3+\tau$ such that if $t>T$,

$$\begin{aligned}
y_1(t)&\geqslant\frac{\alpha_1^L m_1}{\gamma_1^M}(1-\mathrm{e}^{-\gamma_1^M\tau_1})>0,\\
y_2(t)&\geqslant\frac{\alpha_2^L m_1 m_2}{\gamma_2^M}(1-\mathrm{e}^{-\gamma_2^M\tau_2})>0.
\end{aligned}$$

This completes the proof. □

3. Existence of positive periodic solutions

In this section, based on Gaines and Mawhin's continuation theorem of coincidence degree theory we discuss the existence of positive periodic solutions to system (1.2) with initial conditions (1.3)–(1.4). For convenience, we shall summarize in the following a few concepts and results from [6] that will be used in this section.

Let X, Y be real Banach spaces, L: $\mathrm{Dom}\, L\subset X\to Y$ a linear mapping, and $N: X\to Y$ a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L=\operatorname{codim} \operatorname{Im} L<+\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P: X\to X$, and $Q: Y\to Y$ such that $\operatorname{Im} P=\operatorname{Ker} L$, $\operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q)$, then the restriction L_P of L to $\mathrm{Dom}\, L\cap \operatorname{Ker} P: (I-P)X\to \operatorname{Im} L$ is invertible. Denote the inverse of L_P by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N:\bar{\Omega}\to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J: \operatorname{Im} Q\to \operatorname{Ker} L$.

Lemma 3.1. *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(a) *for each* $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda N x$;
(b) *for each* $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.

We are now in a position to state and prove our result on the existence of positive periodic solutions of system (1.2).

Theorem 3.1. *Let* (H1) *hold. Then system* (1.2) *with initial conditions* (1.3) *and* (1.4) *has at least one strictly positive ω-periodic solution.*

Proof. We first consider the following subsystem

$$\begin{aligned} \dot{x}_1(t) &= \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\,\mathrm{d}s}x_1(t-\tau_1) - a_{11}(t)x_1^2(t) - a_{12}(t)x_1(t)x_2(t), \\ \dot{x}_2(t) &= \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\,\mathrm{d}s}x_1(t-\tau_2)x_2(t-\tau_2) - r(t)x_2(t) - a_{22}(t)x_2^2(t). \end{aligned} \tag{3.1}$$

Let

$$u_1(t) = \ln[x_1(t)], \quad u_2(t) = \ln[x_2(t)]. \tag{3.2}$$

On substituting (3.2) into (3.1), we derive that

$$\begin{aligned} \frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - a_{11}(t)\mathrm{e}^{u_1(t)} - a_{12}(t)\mathrm{e}^{u_2(t)}, \\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)} - r(t) - a_{22}(t)\mathrm{e}^{u_2(t)}. \end{aligned} \tag{3.3}$$

It is easy to see that if system (3.3) has one ω-periodic solution $(u_1^*(t), u_2^*(t))^{\mathrm{T}}$, then $(x_1^*(t), x_2^*(t))^{\mathrm{T}} = (\exp[u_1^*(t)], \exp[u_2^*(t)])^{\mathrm{T}}$ is a positive ω-periodic solution of system (3.1). Therefore, in the following we first prove that system (3.3) has at least one ω-periodic solution.

To apply Lemma 3.1 to (3.3), we define

$$X = Y = \{(u_1(t), u_2(t))^{\mathrm{T}} \in C(R, R^2) : u_i(t+\omega) = u_i(t), i = 1, 2\}$$

and

$$\|(u_1(t), u_2(t))^{\mathrm{T}}\| = \max_{t\in[0,\omega]} |u_1(t)| + \max_{t\in[0,\omega]} |u_2(t)|,$$

here $|\cdot|$ denotes the Euclidean norm. Then it is easy to see that both X and Y are Banach spaces with the norm $\|\cdot\|$. Let

$$L : \operatorname{Dom} L \cap X \to X, \quad L(u_1(t), u_2(t))^{\mathrm{T}} = \left(\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}u_2(t)}{\mathrm{d}t}\right)^{\mathrm{T}},$$

where $\operatorname{Dom} L = \{(u_1(t), u_2(t))^{\mathrm{T}} \in C^1(R, R^2)\}$ and $N : X \to X$,

$$N\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^{t}\gamma_1(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - a_{11}(t)\mathrm{e}^{u_1(t)} - a_{12}(t)\mathrm{e}^{u_2(t)} \\ \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^{t}\gamma_2(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)} - r(t) - a_{22}(t)\mathrm{e}^{u_2(t)} \end{bmatrix}.$$

Define

$$P\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Q\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\int_0^{\omega} u_1(t)\,\mathrm{d}t \\ \frac{1}{\omega}\int_0^{\omega} u_2(t)\,\mathrm{d}t \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in X = Y.$$

It is not difficult to show that

$$\operatorname{Ker} L = \{x | x \in X, x = h, h \in R^2\},$$
$$\operatorname{Im} L = \left\{ y | y \in Y, \int_0^\omega y(t)\,\mathrm{d}t = 0 \right\}$$

is closed in Y, and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2,$$

and P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the inverse K_P of L_P exists and is given by $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$,

$$K_P(y) = \int_0^t y(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t y(s)\,\mathrm{d}s\,\mathrm{d}t$$

and $QN : X \to Y$ and $K_P(I - Q)N : X \to X$ are defined by

$$QNx = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega [\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - a_{11}(t)\mathrm{e}^{u_1(t)} - a_{12}(t)\mathrm{e}^{u_2(t)}]\,\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega [\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)} - r(t) - a_{22}(t)\mathrm{e}^{u_2(t)}]\,\mathrm{d}t \end{bmatrix},$$

$$K_P(I-Q)Nx = \int_0^t Nx(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t Nx(s)\,\mathrm{d}s\,\mathrm{d}t - \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^\omega Nx(s)\,\mathrm{d}s.$$

Clearly, QN and $K_P(I - Q)N$ are continuous.

In order to apply Lemma 3.1, we need to search for an appropriate open, bounded subset Ω.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} \frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= \lambda[\alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} - a_{11}(t)\mathrm{e}^{u_1(t)} - a_{12}(t)\mathrm{e}^{u_2(t)}], \\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= \lambda[\alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\,\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_2)+u_2(t-\tau_2)-u_2(t)} - r(t) - a_{22}(t)\mathrm{e}^{u_2(t)}]. \end{aligned} \tag{3.4}$$

Suppose that $(u_1(t), u_2(t))^{\mathrm{T}} \in X$ is a solution of (3.4) for a certain $\lambda \in (0, 1)$.

Since $(u_1(t), u_2(t))^{\mathrm{T}} \in X$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_1(\xi_1) = \max_{t\in[0,\omega]} u_1(t), \quad u_2(\xi_2) = \max_{t\in[0,\omega]} u_2(t) \tag{3.5}$$

and

$$u_1(\eta_1) = \min_{t\in[0,\omega]} u_1(t), \quad u_2(\eta_2) = \min_{t\in[0,\omega]} u_2(t). \tag{3.6}$$

It follows from (3.5) that

$$u_1'(\xi_1) = 0, \quad u_2'(\xi_2) = 0,$$

that is

$$\begin{aligned} &\alpha_1(\xi_1-\tau_1)\mathrm{e}^{-\int_{\xi_1-\tau_1}^{\xi_1} \gamma_1(s)\,\mathrm{d}s}\mathrm{e}^{u_1(\xi_1-\tau_1)-u_1(\xi_1)} - a_{11}(\xi_1)\mathrm{e}^{u_1(\xi_1)} - a_{12}(\xi_1)\mathrm{e}^{u_2(\xi_1)} = 0, \\ &\alpha_2(\xi_2-\tau_2)\mathrm{e}^{-\int_{\xi_2-\tau_2}^{\xi_2} \gamma_2(s)\,\mathrm{d}s}\mathrm{e}^{u_1(\xi_2-\tau_2)+u_2(\xi_2-\tau_2)-u_2(\xi_2)} - r(\xi_2) - a_{22}(\xi_2)\mathrm{e}^{u_2(\xi_2)} = 0. \end{aligned} \tag{3.7}$$

We derive from the first equation of (3.7) that

$$a_{11}(\xi_1)e^{u_1(\xi_1)} \leqslant \alpha_1(\xi_1-\tau_1)e^{-\int_{\xi_1-\tau_1}^{\xi_1}\gamma_1(s)\,ds}e^{u_1(\xi_1-\tau_1)-u_1(\xi_1)} \leqslant \alpha_1^M e^{-\gamma_1^L\tau},$$

which yields

$$u_1(\xi_1) \leqslant \ln\frac{\alpha_1^M e^{-\gamma_1^L\tau}}{a_{11}^L}. \tag{3.8}$$

It follows from the second equation of (3.7) and (3.8) that

$$\begin{aligned} a_{22}(\xi_2)e^{u_2(\xi_2)} &= \alpha_2(\xi_2-\tau_2)e^{-\int_{\xi_2-\tau_2}^{\xi_2}\gamma_2(s)\,ds}e^{u_1(\xi_2-\tau_2)+u_2(\xi_2-\tau_2)-u_2(\xi_2)} - r(\xi_2)\\ &\leqslant \alpha_2^M e^{-\gamma_2^L\tau_2}e^{u_1(\xi_2-\tau_2)} - r^L\\ &\leqslant \frac{\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2}}{a_{11}^L} - r^L, \end{aligned}$$

which yields

$$u_2(\xi_2) \leqslant \ln\frac{\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2} - r^L a_{11}^L}{a_{11}^L a_{22}^L}. \tag{3.9}$$

We derive from (3.6) that

$$u_1'(\eta_1)=0, \qquad u_2'(\eta_2)=0,$$

that is

$$\begin{aligned} &\alpha_1(\eta_1-\tau_1)e^{-\int_{\eta_1-\tau_1}^{\eta_1}\gamma_1(s)\,ds}e^{u_1(\eta_1-\tau_1)-u_1(\eta_1)} - a_{11}(\eta_1)e^{u_1(\eta_1)} - a_{12}(\eta_1)e^{u_2(\eta_1)} = 0,\\ &\alpha_2(\eta_2-\tau_2)e^{-\int_{\eta_2-\tau_2}^{\eta_2}\gamma_2(s)\,ds}e^{u_1(\eta_2-\tau_2)+u_2(\eta_2-\tau_2)-u_2(\eta_2)} - r(\eta_2) - a_{22}(\eta_2)e^{u_2(\eta_2)} = 0. \end{aligned} \tag{3.10}$$

It follows from the first equation of (3.10) and (3.9) that

$$\begin{aligned} a_{11}(\eta_1)e^{u_1(\eta_1)} &\geqslant \alpha_1^L e^{-\gamma_1^M\tau_1}e^{u_1(\eta_1-\tau_1)-u_1(\eta_1)} - a_{12}^M e^{u_2(\eta_1)}\\ &\geqslant \alpha_1^L e^{-\gamma_1^M\tau_1} - a_{12}^M\frac{\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2} - r^L a_{11}^L}{a_{11}^L a_{22}^L}, \end{aligned}$$

which gives

$$u_1(\eta_1) \geqslant \ln\frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - a_{12}^M((\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2} - r^L a_{11}^L)/a_{11}^L a_{22}^L)}{a_{11}^M}. \tag{3.11}$$

We derive from (3.8) and (3.11) that

$$\begin{aligned} \max_{t\in[0,\omega]}|u_1(t)| < \max\Bigg\{&\left|\ln\frac{\alpha_1^M e^{-\gamma_1^L\tau_1}}{a_{11}^L}\right|,\\ &\left|\ln\frac{\alpha_1^L e^{-\gamma_1^M\tau_1} - a_{12}^M((\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2} - r^L a_{11}^L)/a_{11}^L a_{22}^L)}{a_{11}^M}\right|\Bigg\} := R_1. \end{aligned} \tag{3.12}$$

It follows from the second equation of (3.10) that

$$\begin{aligned} a_{22}(\eta_2)e^{u_2(\eta_2)} &\geqslant \alpha_2^L e^{-\gamma_2^M\tau_2} e^{u_1(\eta_2-\tau_2)+u_2(\eta_2-\tau_2)-u_2(\eta_2)} - r^M \\ &\geqslant \alpha_2^L e^{-\gamma_2^M\tau_2} e^{u_1(\eta_2-\tau_2)} - r^M \\ &\geqslant \alpha_2^L e^{-\gamma_2^M\tau_2} e^{u_1(\eta_1)} - r^M \end{aligned}$$

which, together with (3.11), yields

$$u_2(\eta_2) \geqslant \ln \frac{\alpha_1^L\alpha_2^L e^{-\gamma_1^M\tau_1-\gamma_2^M\tau_2} - r^M a_{11}^M - (a_{12}^M\alpha_2^L e^{-\gamma_2^M\tau_2}/a_{11}^L a_{22}^L)(\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2} - r^L a_{11}^L)}{a_{11}^M a_{22}^M} := \rho. \tag{3.13}$$

We obtain from (3.9) and (3.13) that

$$\max_{t\in[0,\omega]} |u_2(t)| < \max\left\{\left|\ln \frac{\alpha_1^M\alpha_2^M e^{-\gamma_1^L\tau_1-\gamma_2^L\tau_2} - r^L a_{11}^L}{a_{11}^L a_{22}^L}\right|, |\rho|\right\} := R_2. \tag{3.14}$$

Clearly, R_1 and R_2 in (3.12) and (3.14) are independent of λ. Denote $M = R_1 + R_2 + R_0$, where R_0 is taken sufficiently large such that the unique solution $(u^*, v^*)^{\mathrm{T}}$ of the system of algebraic equations

$$\begin{aligned} &\frac{1}{\omega}\int_0^\omega \alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^t \gamma_1(s)\,ds}\,dt - \overline{a}_{11}e^u - \overline{a}_{12}e^v = 0, \\ &\frac{1}{\omega}\int_0^\omega \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^t \gamma_2(s)\,ds}\,dt\,e^u - \overline{r} - \overline{a}_{22}e^v = 0 \end{aligned} \tag{3.15}$$

satisfies $\|(u^*, v^*)^{\mathrm{T}}\| = |u^*| + |v^*| < M$.

We now take $\Omega = \{(u_1(t), u_2(t))^{\mathrm{T}} \in X : \|(u_1, u_2)^{\mathrm{T}}\| < M\}$. This satisfies the condition (a) in Lemma 3.1. When $(u_1(t), u_2(t))^{\mathrm{T}} \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $(u_1, u_2)^{\mathrm{T}}$ is a constant vector in R^2 with $|u_1| + |u_2| = M$. Thus, we have

$$QN\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega \alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^t \gamma_1(s)\,ds}\,dt - \overline{a}_{11}e^{u_1} - \overline{a}_{12}e^{u_2} \\ \frac{1}{\omega}\int_0^\omega \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^t \gamma_2(s)\,ds}\,dt\,e^{u_1} - \overline{r} - \overline{a}_{22}e^{u_2} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This proves that condition (b) in Lemma 3.1 is satisfied.

Taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(u_1, u_2)^{\mathrm{T}} \to (u_1, u_2)^{\mathrm{T}}$, a direct calculation shows that

$$\begin{aligned} &\deg(JQN(u_1, u_2)^{\mathrm{T}}, \Omega \cap \operatorname{Ker} L, (0,0)^{\mathrm{T}}) \\ &= \deg\Bigg(\Bigg(\frac{1}{\omega}\int_0^\omega \alpha_1(t-\tau_1)e^{-\int_{t-\tau_1}^t \gamma_1(s)\,ds}\,dt - \overline{a}_{11}e^{u_1} - \overline{a}_{12}e^{u_2}, \\ &\quad \frac{1}{\omega}\int_0^\omega \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^t \gamma_2(s)\,ds}\,dt\,e^{u_1} - \overline{r} - \overline{a}_{22}e^{u_2}\Bigg)^{\mathrm{T}}, \Omega \cap \operatorname{Ker} L, (0,0)^{\mathrm{T}}\Bigg) \\ &= \operatorname{sgn}\left\{\left(\overline{a}_{11}\overline{a}_{22} + \frac{\overline{a}_{12}}{\omega}\int_0^\omega \alpha_2(t-\tau_2)e^{-\int_{t-\tau_2}^t \gamma_2(s)\,ds}\,dt\right)e^{u^*+v^*}\right\} = 1, \end{aligned}$$

where (u^*, v^*) is the unique solution of (3.15).

Finally, it is easy to show that the set $\{K_P(I-Q)Nx \mid x \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela–Ascoli Theorem, we see that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Consequently, N is L-compact.

By now we have proved that Ω satisfies all the requirements in Lemma 3.1. Hence, (3.3) has at least one ω-periodic solution. Accordingly, system (3.1) has at least one positive ω-periodic solution.

Let $(x_1^*(t), x_2^*(t))^{\mathrm{T}}$ be a positive ω-periodic solution of system (3.1). Then it is easy to verify that

$$y_1^*(t) = \int_{t-\tau_1}^{t} \alpha_1(s) e^{-\int_s^t \gamma_1(v)\,dv} x_1^*(s)\,ds$$

and

$$y_2^*(t) = \int_{t-\tau_2}^{t} \alpha_2(s) e^{-\int_s^t \gamma_2(v)\,dv} x_1^*(s) x_2^*(s)\,ds$$

are also ω-periodic. Thus, $(x_1^*(t), y_1^*(t), x_2^*(t), y_2^*(t))^{\mathrm{T}}$ is a positive ω-periodic solution of system (1.2) with initial conditions (1.3)–(1.4). This completes the proof. $\square$

4. Uniqueness and global stability

In this section, we discuss the uniqueness and global stability of the positive periodic solutions of system (1.2) with initial conditions (1.3)–(1.4). The strategy of proof is to construct an appropriate Lyapunov functional.

Theorem 4.1. *In addition to* (H1), *assume further that*

(H2) $\liminf_{t\to\infty} A_i(t) > 0$,

where

$$\begin{aligned} A_1(t) = {} & a_{11}(t) - a_{21}(t+\tau_2) - \frac{r_1(t)M_1}{m_1^2} \int_{t-\tau_1}^{t} r_1(s)\,ds \\ & - \frac{1}{m_1}(2a_{11}(t)M_1 + a_{12}(t)M_2) \int_{t}^{t+\tau_1} r_1(s)\,ds \\ & - \frac{r_1(t+\tau_1)}{m_1} \int_{t+\tau_1}^{t+2\tau_1} r_1(s)\,ds - \frac{a_{21}(t+\tau_2)M_1M_2}{m_2} \int_{t+\tau_2}^{t+2\tau_2} a_{21}(s)\,ds \\ & - \frac{a_{21}(t+\tau_2)M_2}{m_2} \int_{t}^{t+\tau_2} [a_{21}(s)M_1 + r(s) + a_{22}(s)M_2]\,ds, \end{aligned}$$

$$\begin{aligned} A_2(t) = {} & a_{22}(t) - a_{12}(t) - \frac{a_{21}(t)M_1^2M_2}{m_2^2} \int_{t-\tau_2}^{t} a_{21}(s)\,ds \\ & - \frac{M_1}{m_2}(r(t) + 2a_{22}(t)M_2) \int_{t}^{t+\tau_2} a_{21}(s)\,ds \\ & - \frac{a_{12}(t)M_1}{m_1} \int_{t}^{t+\tau_1} r_1(s)\,ds - \frac{a_{21}(t+\tau_2)M_1^2}{m_2} \int_{t+\tau_2}^{t+2\tau_2} a_{21}(s)\,ds \end{aligned} \tag{4.1}$$

in which $r_1(t) = \alpha_1(t-\tau_1) e^{-\int_{t-\tau_1}^{t} \gamma_1(s)\,ds}$, $a_{21}(t) = \alpha_2(t-\tau_2) e^{-\int_{t-\tau_2}^{t} \gamma_2(s)\,ds}$, *$m_i$ and M_i are defined in* (2.2)–(2.5). *Then system* (1.2) *has a unique positive ω-periodic solution $(x_1^*(t), y_1^*(t), x_2^*(t), y_2^*(t))^{\mathrm{T}}$ which is globally stable.*

Proof. By Theorem 3.1 we see that if (H1) holds, system (1.2) has at least one positive periodic solution. Therefore, it suffices to show the global stability of positive periodic solutions of system (1.2) with initial conditions (1.3) and (1.4). Let $(x_1^*(t), y_1^*(t), x_2^*(t), y_2^*(t))^{\mathrm{T}}$ be a positive ω-periodic solution of system (1.2) with initial conditions (1.3)–(1.4).

Suppose that $(x_1(t), y_1(t), x_2(t), y_2(t))^{\mathrm{T}}$ is a positive solution of system (1.2) with initial conditions

$$\begin{aligned}
&x_i(\theta) = \Phi_i(\theta), \quad y_i(\theta) = \Psi_i(\theta),\\
&\Phi_i(0) > 0, \quad \Psi_i(0) > 0, \quad i = 1, 2,\\
&(\Phi_1(\theta), \Psi_1(\theta), \Phi_2(\theta), \Psi_2(\theta)) \in C([-\tau, 0], R^4_{+0})
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
y_1(0) &= \int_{-\tau_1}^{0} \alpha_1(s) \mathrm{e}^{-\int_s^0 \gamma_1(v)\,\mathrm{d}v} \Phi_1(s)\,\mathrm{d}s,\\
y_2(0) &= \int_{-\tau_2}^{0} \alpha_2 \mathrm{e}^{-\int_s^0 \gamma_2(v)\,\mathrm{d}v} \Phi_1(s)\Phi_2(s)\,\mathrm{d}s.
\end{aligned} \tag{4.3}$$

We first consider the following subsystem of system (1.2)

$$\begin{aligned}
\dot{x}_1(t) &= r_1(t)x_1(t-\tau_1) - a_{11}(t)x_1^2(t) - a_{12}(t)x_1(t)x_2(t),\\
\dot{x}_2(t) &= a_{21}(t)x_1(t-\tau_2)x_2(t-\tau_2) - r(t)x_2(t) - a_{22}(t)x_2^2(t),
\end{aligned} \tag{4.4}$$

where $r_1(t) = \alpha_1(t-\tau_1)\mathrm{e}^{-\int_{t-\tau_1}^t \gamma_1(s)\,\mathrm{d}s}$, $a_{21}(t) = \alpha_2(t-\tau_2)\mathrm{e}^{-\int_{t-\tau_2}^t \gamma_2(s)\,\mathrm{d}s}$.

We rewrite system (4.4) as follows

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t)]\\
&\quad - r_1(t)\int_{t-\tau_1}^{t} \{r_1(u)x_1(u-\tau_1) - a_{11}(u)x_1^2(u) - a_{12}(u)x_1(u)x_2(u)\}\,\mathrm{d}u,\\
\dot{x}_2(t) &= x_2(t)[-r(t) + a_{21}(t)x_1(t-\tau_2) - a_{22}(t)x_2(t)]\\
&\quad - a_{21}(t)x_1(t-\tau_2)\int_{t-\tau_2}^{t} \{a_{21}(u)x_1(u-\tau_2)x_2(u-\tau_2)\\
&\quad - r(u)x_2(u) - a_{22}(u)x_2^2(u)\}\,\mathrm{d}u.
\end{aligned} \tag{4.5}$$

Let

$$V_{11}(t) = |\ln x_1(t) - \ln x_1^*(t)|. \tag{4.6}$$

Calculating the upper right derivative of $V_{11}(t)$ along positive solutions of (4.5), it follows that

$$\begin{aligned}
D^+V_{11}(t) &= \left(\frac{\dot{x}_1(t)}{x_1(t)} - \frac{\dot{x}_1^*(t)}{x_1^*(t)}\right) \operatorname{sgn}(x_1(t) - x_1^*(t))\\
&= \operatorname{sgn}(x_1(t) - x_1^*(t)) \Bigg\{ - a_{11}(t)(x_1(t) - x_1^*(t)) - a_{12}(t)(x_2(t) - x_2^*(t))\\
&\quad - \frac{r_1(t)}{x_1(t)} \int_{t-\tau_1}^{t} \{r_1(u)x_1(u-\tau_1) - a_{11}(u)x_1^2(u) - a_{12}(u)x_1(u)x_2(u)\}\,\mathrm{d}u\\
&\quad + \frac{r_1(t)}{x_1^*(t)} \int_{t-\tau_1}^{t} \{r_1(u)x_1^*(u-\tau_1) - a_{11}(u)x_1^{*2}(u) - a_{12}(u)x_1^*(u)x_2^*(u)\}\,\mathrm{d}u \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{sgn}(x_1(t) - x_1^*(t)) \Bigg\{ - a_{11}(t)(x_1(t) - x_1^*(t)) - a_{12}(t)(x_2(t) - x_2^*(t)) \\
&\quad - \frac{r_1(t)}{x_1(t)} \int_{t-\tau_1}^{t} \{r_1(u)(x_1(u-\tau_1) - x_1^*(u-\tau_1)) \\
&\quad - a_{11}(u)(x_1(u) + x_1^*(u))(x_1(u) - x_1^*(u)) \\
&\quad - a_{12}(u)[x_2(u)(x_1(u) - x_1^*(u)) + x_1^*(u)(x_2(u) - x_2^*(u))]\} \, \mathrm{d}u \\
&\quad + \frac{r_1(t)(x_1(t) - x_1^*(t))}{x_1(t)x_1^*(t)} \int_{t-\tau_1}^{t} \{r_1(u)x_1^*(u-\tau_1) \\
&\quad - a_{11}(u)x_1^{*2}(u) - a_{12}(u)x_1^*(u)x_2^*(u)\} \, \mathrm{d}u \Bigg\} \\
&\leqslant - a_{11}(t)|x_1(t) - x_1^*(t)| + a_{12}(t)|x_2(t) - x_2^*(t)| \\
&\quad + \frac{r_1(t)}{x_1(t)} \int_{t-\tau_1}^{t} \{r_1(u)|x_1(u-\tau_1) - x_1^*(u-\tau_1)| \\
&\quad + a_{11}(u)(x_1(u) + x_1^*(u))|x_1(u) - x_1^*(u)| \\
&\quad + a_{12}(u)[x_2(u)|x_1(u) - x_1^*(u)| + x_1^*(u)|x_2(u) - x_2^*(u)|]\} \, \mathrm{d}u \\
&\quad + \frac{r_1(t)|x_1(t) - x_1^*(t)|}{x_1(t)x_1^*(t)} \int_{t-\tau_1}^{t} r_1(u)x_1^*(u-\tau_1) \, \mathrm{d}u.
\end{aligned} \tag{4.7}
$$

By Theorem 2.1 there exists a $T > 0$ such that if $t > T$,

$$
m_i < x_i(t) < M_i, \quad m_i < x_i^*(t) < M_i, \quad i = 1, 2, \tag{4.8}
$$

where m_i and M_i are defined in (2.2)–(2.5). We derive from (4.7) and (4.8) that for $t > T + 2\tau$,

$$
\begin{aligned}
D^+V_{11}(t) \leqslant & - a_{11}(t)|x_1(t) - x_1^*(t)| + a_{12}(t)|x_2(t) - x_2^*(t)| \\
& + \frac{r_1(t)}{m_1} \int_{t-\tau_1}^{t} \{r_1(u)|x_1(u-\tau_1) - x_1^*(u-\tau_1)| \\
& + (2a_{11}(u)M_1 + a_{12}(u)M_2)|x_1(u) - x_1^*(u)| \\
& + a_{12}(u)M_1|x_2(u) - x_2^*(u)|\} \, \mathrm{d}u \\
& + \frac{r_1(t)M_1}{m_1^2} \int_{t-\tau_1}^{t} r_1(u) \, \mathrm{d}u |x_1(t) - x_1^*(t)|.
\end{aligned} \tag{4.9}
$$

Define

$$
\begin{aligned}
V_{12}(t) = & \frac{1}{m_1} \int_{t}^{t+\tau_1} \int_{s-\tau_1}^{t} r_1(s)\{r_1(u)|x_1(u-\tau_1) - x_1^*(u-\tau_1)| \\
& + (2a_{11}(u)M_1 + a_{12}(u)M_2)|x_1(u) - x_1^*(u)| \\
& + a_{12}(u)M_1|x_2(u) - x_2^*(u)|\} \, \mathrm{d}u \, \mathrm{d}s.
\end{aligned} \tag{4.10}
$$

Then it follows from (4.9) and (4.10) that for $t > T + 2\tau$,

$$\begin{aligned}D^+V_{11}(t) + \dot{V}_{12}(t) \leqslant & -a_{11}(t)|x_1(t) - x_1^*(t)| + a_{12}(t)|x_2(t) - x_2^*(t)| \\ & + \frac{1}{m_1}\int_t^{t+\tau_1} r_1(s)\,\mathrm{d}s\{r_1(t)|x_1(t-\tau_1) - x_1^*(t-\tau_1)| \\ & + (2a_{11}(t)M_1 + a_{12}(t)M_2)|x_1(t) - x_1^*(t)| \\ & + a_{12}(t)M_1|x_2(t) - x_2^*(t)|\} \\ & + \frac{r_1(t)M_1}{m_1^2}\int_{t-\tau_1}^{t} r_1(u)\,\mathrm{d}u|x_1(t) - x_1^*(t)|. \end{aligned} \tag{4.11}$$

Let

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t), \tag{4.12}$$

where

$$V_{13}(t) = \frac{1}{m_1}\int_{t-\tau_1}^{t}\int_{s+\tau_1}^{s+2\tau_1} r_1(s+\tau_1)r_1(u)|x_1(s) - x_1^*(s)|\,\mathrm{d}u\,\mathrm{d}s. \tag{4.13}$$

We derive from (4.11)–(4.13) that for $t > T + 2\tau$,

$$\begin{aligned}D^+V_1(t) \leqslant & -a_{11}(t)|x_1(t) - x_1^*(t)| + a_{12}(t)|x_2(t) - x_2^*(t)| \\ & + \frac{1}{m_1}\int_t^{t+\tau_1} r_1(s)\,\mathrm{d}s\{a_{12}(t)M_1|x_2(t) - x_2^*(t)| \\ & + (2a_{11}(t)M_1 + a_{12}(t)M_2)|x_1(t) - x_1^*(t)|\} \\ & + \frac{1}{m_1}r_1(t+\tau_1)\int_{t+\tau_1}^{t+2\tau_1} r_1(s)\,\mathrm{d}s|x_1(t) - x_1^*(t)| \\ & + \frac{r_1(t)M_1}{m_1^2}\int_{t-\tau_1}^{t} r_1(s)\,\mathrm{d}s|x_1(t) - x_1^*(t)|. \end{aligned} \tag{4.14}$$

Similarly, we define

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t), \tag{4.15}$$

where

$$\begin{aligned}V_{21}(t) = & |\ln x_2(t) - \ln x_2^*(t)|, \\ V_{22}(t) = & \int_{t-\tau_2}^{t} a_{21}(s+\tau_2)|x_1(s) - x_1^*(s)|\,\mathrm{d}s \\ & + \frac{M_1}{m_2}\int_t^{t+\tau_2}\int_{s-\tau_2}^{t} a_{21}(s)\{a_{21}(u)[M_2|x_1(u-\tau_2) - x_1^*(u-\tau_2)| \\ & + M_1|x_2(u-\tau_2) - x_2^*(u-\tau_2)|] + (2a_{22}(u)M_2 + r(u))|x_2(u) - x_2^*(u)|\}\,\mathrm{d}u\,\mathrm{d}s \\ & + \frac{M_2}{m_2}\int_{t-\tau_2}^{t}\int_{s}^{s+\tau_2} a_{21}(s+\tau_2)[a_{21}(u)M_1 + r(u) + a_{22}(u)M_2]|x_1(s) - x_1^*(s)|\,\mathrm{d}u\,\mathrm{d}s, \\ V_{23}(t) = & \frac{M_1}{m_2}\int_{t-\tau_2}^{t}\int_{s+\tau_2}^{s+2\tau_2} a_{21}(s+\tau_2)a_{21}(u)[M_2|x_1(s) - x_1^*(s)| \\ & + M_1|x_2(s) - x_2^*(s)|]\,\mathrm{d}u\,\mathrm{d}s. \end{aligned} \tag{4.16}$$

Calculating the upper right derivative of $V_2(t)$ along positive solutions of (4.5), it follows from (4.15) and (4.16) that for $t > T + 2\tau$,

$$\begin{aligned} D^+V_2(t) \leqslant & -a_{22}(t)|x_2(t) - x_2^*(t)| + a_{21}(t+\tau_2)|x_1(t) - x_1^*(t)| \\ & + \frac{M_1}{m_2}\int_t^{t+\tau_2} a_{21}(s)\,\mathrm{d}s(r(t) + 2a_{22}(t)M_2)|x_2(t) - x_2^*(t)| \\ & + \frac{a_{21}(t+\tau_2)M_1}{m_2}\int_{t+\tau_2}^{t+2\tau_2} a_{21}(s)\,\mathrm{d}s[M_2|x_1(t) - x_1^*(t)| + M_1|x_2(t) - x_2^*(t)|] \\ & + \frac{a_{21}(t)M_1^2M_2}{m_2^2}\int_{t-\tau_2}^{t} a_{21}(u)\,\mathrm{d}u|x_2(t) - x_2^*(t)| + \frac{a_{21}(t+\tau_2)M_2}{m_2}|x_1(t) - x_1^*(t)| \\ & \times \int_t^{t+\tau_2}[a_{21}(s)M_1 + r(s) + a_{22}(s)M_2]\,\mathrm{d}s. \end{aligned} \tag{4.17}$$

We now define

$$V(t) = V_1(t) + V_2(t). \tag{4.18}$$

Then it follows from (4.14), (4.17) and (4.18) that for $t \geqslant T + 2\tau$

$$D^+V(t) \leqslant -A_1(t)|x_1(t) - x_1^*(t)| - A_2(t)|x_2(t) - x_2^*(t)|, \tag{4.19}$$

where $A_1(t)$ and $A_2(t)$ are defined in (4.1).

By the assumption (H2), there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ and a $T^* \geqslant T + 2\tau$, such that for $t \geqslant T^*$,

$$A_1(t) \geqslant \alpha_1 > 0, \quad A_2(t) \geqslant \alpha_2 > 0. \tag{4.20}$$

Integrating both sides of (4.19) on interval $[T^*, t]$, it follows that for $t \geqslant T^*$

$$V(t) + \int_{T^*}^t A_1(s)|x_1(s) - x_1^*(s)|\,\mathrm{d}s + \int_{T^*}^t A_2(s)|x_2(s) - x_2^*(s)|\,\mathrm{d}s \leqslant V(T^*). \tag{4.21}$$

We derive from (4.20) and (4.21) that

$$V(t) + \alpha_1\int_{T^*}^t |x_1(s) - x_1^*(s)|\,\mathrm{d}s + \alpha_2\int_{T^*}^t |x_2(s) - x_2^*(s)|\,\mathrm{d}s \leqslant V(T^*) \quad \text{for } t \geqslant T^*.$$

Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^\infty |x_1(s) - x_1^*(s)|\,\mathrm{d}s < \infty, \quad \int_{T^*}^\infty |x_2(s) - x_2^*(s)|\,\mathrm{d}s < \infty.$$

By Theorem 2.1, $|x_1(t) - x_1^*(t)|$ and $|x_2(t) - x_2^*(t)|$ are bounded on $[T^*, \infty)$.

On the other hand, it is easy to see that $\dot{x}_1(t)$, $\dot{x}_1^*(t)$, $\dot{x}_2(t)$ and $\dot{x}_2^*(t)$ are bounded for $t \geqslant T^*$. Therefore, $|x_1(t) - x_1^*(t)|$ and $|x_2(t) - x_2^*(t)|$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's Lemma [7, Lemmas 1.2.2 and 1.2.3], we conclude that

$$\lim_{t\to\infty} |x_1(t) - x_1^*(t)| = 0, \quad \lim_{t\to\infty} |x_2(t) - x_2^*(t)| = 0. \tag{4.22}$$

It follows from (2.1) that

$$\begin{aligned} y_1(t) &= \int_{t-\tau_1}^t \alpha_1(s)\mathrm{e}^{-\int_s^t \gamma_1(v)\,\mathrm{d}v}x_1(s)\,\mathrm{d}s, \\ y_1^*(t) &= \int_{t-\tau_1}^t \alpha_1(s)\mathrm{e}^{-\int_s^t \gamma_1(v)\,\mathrm{d}v}x_1^*(s)\,\mathrm{d}s. \end{aligned} \tag{4.23}$$

We obtain from (4.22) that for $\forall \varepsilon > 0$, there exists a $T_1^* > 0$ such that if $t > T_1^*$,

$$|x_1(t) - x_1^*(t)| < \frac{\varepsilon}{2\alpha_1^M}. \tag{4.24}$$

Thus, it follows from (4.23) and (4.24) that for $t > T_1^* + \tau$,

$$\begin{aligned}|y_1(t) - y_1^*(t)| &\leqslant \int_{t-\tau_1}^{t} \alpha_1(s) \mathrm{e}^{-\int_s^t \gamma_1(v)\,\mathrm{d}v} |x_1(s) - x_1^*(s)|\,\mathrm{d}s \\ &\leqslant \int_{t-\tau}^{t} \alpha_1^M \mathrm{e}^{-\gamma_1^L (t-s)} \frac{\varepsilon}{2\alpha_1^M}\,\mathrm{d}s < \varepsilon.\end{aligned}$$

We therefore obtain

$$\lim_{t\to\infty} |y_1(t) - y_1^*(t)| = 0.$$

Similarly, we can derive from (2.1) and (4.22) that

$$\lim_{t\to+\infty} |y_2(t) - y_2^*(t)| = 0.$$

The proof is complete. □

5. Discussion

In this paper, based on the recent work of Aiello and Freedman [1], we incorporated the periodicity of ecological and environmental parameters, stage structure for both the prey and predator populations into a Lotka–Volterra type predator–prey model. To the best of our knowledge, this is the first time to consider system (1.2). By several comparison arguments we have derived a result on the permanence of system (1.2). By using Gaines and Mawhin's continuation theorem of coincidence degree theory, sufficient conditions are derived for the existence of positive periodic solutions to system (1.2). By constructing a suitable Lyapunov functional, sufficient conditions are obtained to guarantee that system (1.2) has a unique positive periodic solution which is globally attractive. By Theorems 2.1 and 3.1 we see that system (1.2) with initial conditions (1.3) and (1.4) is permanent and admits at least one positive periodic solution if the transformation rates from the immature stage to the mature stage for both prey and predator populations dominate the intra-specific competition rate of the prey and the death rate of the mature predator, and the intra-specific competition rates of prey and predator dominate the capturing rate and conversion rate of the predator satisfying (H1). By Theorem 4.1 we see that the positive periodic solution is unique and globally stable provided that the intra-specific competition of prey and predator dominate the inter-specific interaction between the prey and the predator and the time delays τ_1 and τ_2 are sufficiently small.

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